18-202: Mathematical Foundations of Electrical Engineering, Fall 2013
Homework 4, due Monday, October 7

1 Section 2.7 Problem 11 [12 Points]

Note: for problems 1-4, do the following:

(a) [**4 Points**] Solve the equation for the particular solution, $y_p(x)$, using the differential equation and right-hand side driving function.

(b) [**4 Points**] Solve the equation for the homogeneous solution, $y_h(x)$, using the differential equation and initial conditions (remember, the initial conditions to use for the homogeneous solution are derived by subtracting the initial condition contribution from the particular solution found in part (a) from the given initial conditions). Sum $y_p(x)$ and $y_h(x)$ to get the total solution, $y(x)$.

(c) [**4 Points**] Check your result, by differentiating your result for $y(x)$ and substituting it back into the differential equation and showing that it solves the equation. Check the initial conditions too.

Solve the initial value problem. Show each step of your calculation in detail.

$$y'' + 3y = 18x^2,\ y(0) = -3,\ y'(0) = 0$$

Solution

(a) We let $y_p(x) = ax^2 + bx + c$. Then,

$$y_p'(x) = 2ax + b$$
$$y_p''(x) = 2a$$

Substituting into the equation, we have

$$2a + 3(ax^2 + bx + c) = 3ax^2 + 3bx + 2a + 3c = 18x^2$$

which gives us the system of equations

$$3a = 18$$
$$3b = 0$$
$$2a + 3c = 0$$

This has the solution $a = 6$, $b = 0$, and $c = -4$, so $y_p(x) = 6x^2 - 4$.
We can also see that $y_p(0) = -4$ and $y_p'(0) = 0$.

(b) We solve the characteristic equation:

$$\lambda^2 + 3 = 0$$
$$\lambda = \frac{\pm\sqrt{-12}}{2} = \pm j\sqrt{3}.$$

Therefore, the homogeneous solution is of the form $y_h(x) = k_1 \cos(x\sqrt{3}) + k_2 \sin(x\sqrt{3})$.

We know from our initial conditions and from the contribution of the particular solution at $x = 0$ that $y_h(0) = 1$ and $y_h'(0) = 0$. Since $y_h(0) = k_1$ and $y_h'(0) = k_2\sqrt{3}$, we know that $k_1 = 1$ and $k_2 = 0$. Therefore, our solution is

$$y(x) = \cos(x\sqrt{3}) + 6x^2 - 4.$$

(c) We take the first and second derivatives of y:

$$\begin{aligned} y'(x) &= -\sqrt{3}\sin(x\sqrt{3}) + 12x \\ y''(x) &= -3\cos(x\sqrt{3}) + 12 \end{aligned}$$

Substituting this into the equation, we get

$$\begin{aligned} -3\cos(x\sqrt{3}) + 12 + 3(\cos(x\sqrt{3}) + 6x^2 - 4) &= -3\cos(x\sqrt{3}) + 12 + 3\cos(x\sqrt{3}) + 18x^2 - 12 \\ &= 18x^2. \end{aligned}$$

2 Section 2.7 Problem 12 [12 Points]

Solve the initial value problem. Show each step of your calculation in detail.

$$y'' + 4y = -12\sin 2x,\ y(0) = 1.8,\ y'(0) = 5.0$$

Solution

(a) We can see that out characteristic equation will be $\lambda^2 + 4 = 0$, resulting in roots of $\pm 2j$. Therefore, we must use the modification rule to find the particular solution. Then

$$\begin{aligned} y_p(x) &= Kx\cos(2x) + Mx\sin(2x) \\ y_p'(x) &= K\cos(2x) - 2Kx\sin(2x) + M\sin(2x) + 2Mx\cos(2x) \\ y_p''(x) &= -2K\sin(2x) - 2K\sin(2x) - 4Kx\cos(2x) + 2M\cos(2x) + 2M\cos(2x) - 4Mx\sin(2x) \\ &= (-4Kx + 4M)\cos(2x) + (-4K - 4Mx)\sin(2x) \end{aligned}$$

and so we have the following system of equations:

$$\begin{aligned} 4M &= 0 \\ -4K &= -12 \end{aligned}$$

This has the solution $K = 3$, $M = 0$. Therefore, $y_p(x) = 3x\cos(2x)$. The initial condition contribution of the particular solution is

$$\begin{aligned} y_p(0) &= 0 \\ y_p'(0) &= 3. \end{aligned}$$

(b) As we saw above, the roots of our characteristic equation are $\pm 2j$. Therefore, our homogeneous solution has the form $y_h(x) = k_1\cos(2x) + k_2\sin(2x)$. Based on the initial condition contribution of the particular solution,

$$\begin{aligned} y_h(0) &= 1.8 \\ y_h'(0) &= 2. \end{aligned}$$

Thus we have the system of equations

$$\begin{aligned} k_1 &= 1.8 \\ 2k_2 &= 2 \end{aligned}$$

and therefore $k_1 = 1.8$ and $k_2 = 1$. Thus our solution is

$$y(x) = 1.8\cos(2x) + \sin(2x) + 3x\cos(2x).$$

(c) Taking derivatives, we have

$$\begin{aligned} y'(x) &= -3.6\sin(2x) + 2\cos(2x) + 3\cos(2x) - 6x\sin(2x) \\ &= -3.6\sin(2x) + 5\cos(2x) - 6x\sin(2x) \\ y''(x) &= -7.2\cos(2x) - 10\sin(2x) - 6\sin(2x) - 12x\cos(2x) \\ &= -7.2\cos(2x) - 16\sin(2x) - 12x\cos(2x) \end{aligned}$$

Substituting into our equation, we get

$$
\begin{aligned}
-7.2\cos(2x) - 16\sin(2x) - 12x\cos(2x) + 4(1.8\cos(2x) + \sin(2x) + 3x\cos(2x)) \\
&= -7.2\cos(2x) - 16\sin(2x) - 12x\cos(2x) + 7.2\cos(2x) + 4\sin(2x) + 12x\cos(2x) \\
&= -12\sin(2x).
\end{aligned}
$$

3 Chapter 2 Review Problem 19 [12 Points]

Solve the problem, showing the details of your work. Sketch or graph the solution.

$$y'' + 16y = 17e^x,\ y(0) = 6,\ y'(0) = -2$$

Solution

(a) Let $y_p(x) = ke^x$. Then, $y_p'(x) = y_p''(x) = ke^x$. Substituting into our equation, we have

$$17ke^x = 17e^x$$

and thus $k = 1$. Therefore, $y_p(x) = e^x$. Our initial condition contribution is therefore

$$y_p(0) = 1$$
$$y_p'(0) = 1$$

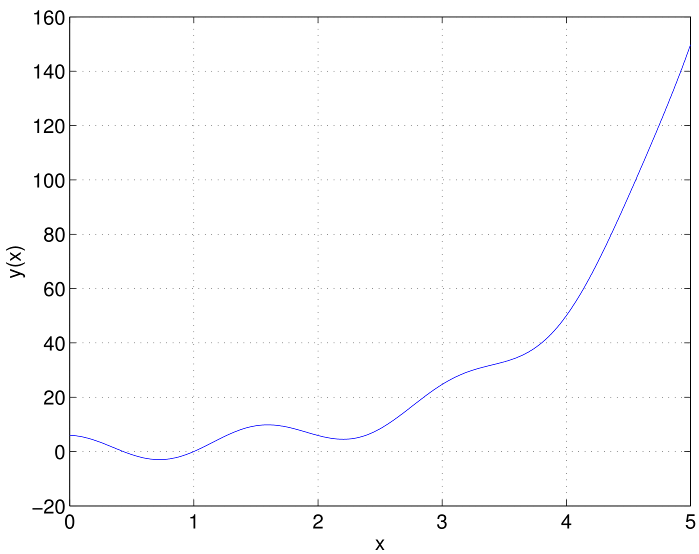


Figure 1: Graph of solution to Problem 3.

(b) Since our characteristic equation is $\lambda^2 + 16 = 0$, our roots are $\lambda = \pm 4j$. Therefore our homogeneous solution takes the form $y_h(x) = k_1 \cos(4x) + k_2 \sin(4x)$. Our initial condition contribution from $y_h(x)$ is

$$y_h(0) = 5$$
$$y_h'(0) = -3.$$

Therefore, we have the system of equations

$$\begin{aligned} k_1 &= 5 \\ 4k_2 &= -3 \end{aligned}$$

which has the solutions $k_1 = 5$, $k_2 = -0.75$. Therefore, our solution is

$$y(x) = 5\cos(4x) - 0.75\sin(4x) + e^x.$$

(c) Taking derivatives, we have

$$\begin{aligned} y'(x) &= -20\sin(4x) - 3\cos(4x) + e^x \\ y''(x) &= -80\cos(4x) + 12\sin(4x) + e^x. \end{aligned}$$

Substituting into the equation, we get

$$\begin{aligned} -80\cos(4x) + 12\sin(4x) + e^x &+ 16(5\cos(4x) - 0.75\sin(4x) + e^x) \\ &= -80\cos(4x) + 12\sin(4x) + e^x + 80\cos(4x) - 12\sin(4x) + 16e^x \\ &= 17e^x. \end{aligned}$$

4 Chapter 2 Review Problem 19 [12 Points]

Solve the problem, showing the details of your work. Sketch or graph the solution.

$$y'' - 3y' + 2y = 10 \sin x, \; y(0) = 1, \; y'(0) = -6$$

Solution

(a) We can solve for the particular solution using the system of equations

$$(c - a\omega^2)K + b\omega M = d$$
$$-b\omega K + (c - a\omega^2)M = e$$

where $a = 1$, $b = -3$, $c = 2$, $d = 0$, $e = 10$, and $\omega = 1$. This gives us the system

$$K - 3M = 0$$
$$3K + M = 10$$

which has the solution $K = 3$, $M = 1$. Therefore $y_p(x) = 3\cos x + \sin x$. The initial condition contribution is

$$y_p(0) = 3$$
$$y_p'(0) = 1$$

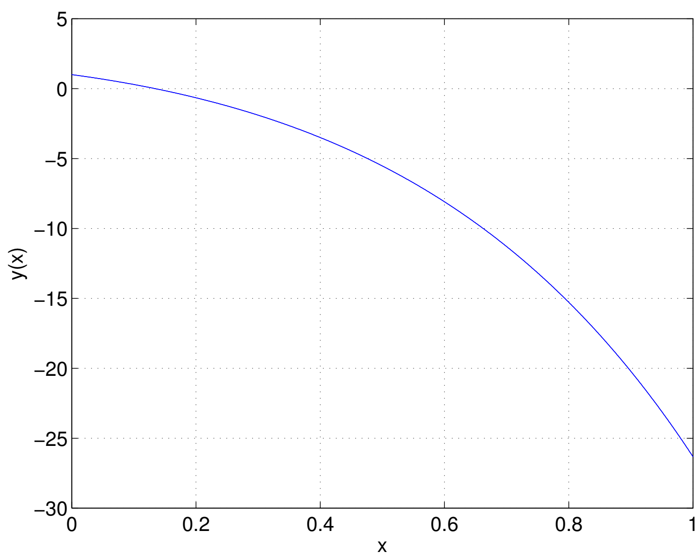


Figure 2: Graph of solution to Problem 4.

(b) Our characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$, which we can factor to get $(\lambda - 1)(\lambda - 2) = 0$. Therefore, our roots are 1 and 2, and thus our homogeneous solution is of the form

$$y_h(x) = k_1 e^x + k_2 e^{2x}.$$

We know that our initial condition contribution is

$$\begin{aligned} y_h(0) &= -2 \\ y_h'(0) &= -7 \end{aligned}$$

This gives us the system of equations

$$\begin{aligned} k_1 + k_2 &= -2 \\ k_1 + 2k_2 &= -7 \end{aligned}$$

which has the solutions $k_1 = 3$, $k_2 = -5$. Thus our final solution is

$$y(x) = 3e^x - 5e^{2x} + 3\cos x + \sin x.$$

(c) Taking derivatives, we have

$$\begin{aligned} y'(x) &= 3e^x - 10e^{2x} - 3\sin x + \cos x \\ y''(x) &= 3e^x - 20e^{2x} - 3\cos x - \sin x \end{aligned}$$

Substituting into our equation, we have

$$\begin{aligned} &3e^x - 20e^{2x} - 3\cos x - \sin x - 3(3e^x - 10e^{2x} - 3\sin x + \cos x) + 2(3e^x - 5e^{2x} + 3\cos x + \sin x) \\ &\quad = (3 - 9 + 6)e^x + (-20 + 30 - 10)e^{2x} + (-3 - 3 + 6)\cos x + (-1 + 9 + 2)\sin x \\ &\quad = 10\sin x \end{aligned}$$

5 Homogeneous RLC Circuit [22 Points]

Solution

(a) Based on KVL, we know that $V_C = V_R + V_L$. Since this means that the polarity of the resistor and inductor is opposite that of the capacitor, KCL gives us $I_C = -I_R = -I_L$. We can take the derivative of the first equation to get $V_C' = V_R' + V_L'$. We know that $I_C = CV_C'$, $V_R = RI_R$, and $V_L = LI_L'$. Substituting into the equation we get

$$I_C/C = RI_R + (LI_L')' = -I_L/C + RI_L + LI_L''.$$

We rewrite this as our governing equation

$$I_L'' + \frac{R}{L}I_L' + \frac{1}{LC}I_L = 0.$$

(b) **Initial condition.** We know that $V_L = LI_L'$. Therefore, $I_L'(0) = V_L(0)/L$. By our KVL equation, $V_L = V_C - V_R = V_C - RI_R = V_C - RI_L$. Therefore,

$$I_L'(0) = V_L(0)/L = (V_C(0) - RI_L(0))/L = (V_0 - 0)/L = V_0/L.$$

Our characteristic equation is

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

which we can solve using the quadratic formula:

$$\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}.$$

For simplicity, let

$$\alpha = -\frac{R}{L}$$
$$\beta = \left(\frac{R}{L}\right)^2 - \frac{4}{LC}.$$

Then, $\lambda = (\alpha \pm \sqrt{\beta})/2$.

Overdamped case. If $\beta > 0$, our system is overdamped. Then, our solution is of the form

$$I_L(t) = k_1 e^{\left(\frac{\alpha - \sqrt{\beta}}{2}\right)t} + k_2 e^{\left(\frac{\alpha + \sqrt{\beta}}{2}\right)t}$$

We then use the initial conditions to obtain the system of equations

$$I_L(0) = k_1 + k_2 = 0$$
$$I_L'(0) = \left(\frac{\alpha - \sqrt{\beta}}{2}\right)k_1 + \left(\frac{\alpha + \sqrt{\beta}}{2}\right)k_2 = \frac{V_0}{L}$$

Thus $k_1 = -k_2$, which gives us

$$-\left(\frac{\alpha - \sqrt{\beta}}{2}\right)k_2 + \left(\frac{\alpha + \sqrt{\beta}}{2}\right)k_2 = \sqrt{\beta}k_2 = \frac{V_0}{L}$$

Therefore, $k_1 = -\frac{V_0}{L\sqrt{\beta}}$ and $k_2 = \frac{V_0}{L\sqrt{\beta}}$. In the denominator,

$$L\sqrt{\beta} = \sqrt{L^2\beta} = \sqrt{R^2 - 4\frac{L}{C}}.$$

Thus

$$\begin{aligned} I_L(t) &= -\frac{V_0}{L\sqrt{\beta}}e^{\left(\frac{\alpha-\sqrt{\beta}}{2}\right)t} + \frac{V_0}{L\sqrt{\beta}}e^{\left(\frac{\alpha+\sqrt{\beta}}{2}\right)t} \\ &= -\frac{V_0}{\sqrt{R^2 - 4\frac{L}{C}}}e^{\left(\frac{-\frac{R}{L}-\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}\right)t} + \frac{V_0}{\sqrt{R^2 - 4\frac{L}{C}}}e^{\left(\frac{-\frac{R}{L}+\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}\right)t} \end{aligned}$$

Critically damped case. If $\beta = 0$, then the system is critically damped. Our solution will be of the form

$$I_L(t) = k_1 e^{\alpha t/2} + k_2 t e^{\alpha t/2}.$$

We can use the initial conditions to obtain the system of equations

$$\begin{aligned} I_L(0) &= k_1 = 0 \\ I_L'(0) &= \alpha k_1/2 + k_2 = \frac{V_0}{L}. \end{aligned}$$

Then, $k_1 = 0$ and $k_2 = V_0/L$. Therefore,

$$I_L(t) = \frac{V_0}{L}te^{\alpha t/2} = \frac{V_0}{L}te^{-\frac{R}{2L}t}.$$

Underdamped case. If $\beta < 0$, then the system is underdamped. Then

$$\lambda = \frac{\alpha \pm j\sqrt{-\beta}}{2}$$

Therefore, our solution is of the form

$$I_L(t) = e^{\alpha t/2}\left(k_1 \cos\left(\sqrt{-\beta}t/2\right) + k_2 \sin\left(\sqrt{-\beta}t/2\right)\right).$$

Using the initial conditions we can obtain the system of equations

$$\begin{aligned} I_L(0) &= k_1 = 0 \\ I_L'(0) &= \alpha k_1/2 + k_1\sqrt{-\beta}/2 = \frac{V_0}{L} \end{aligned}$$

Thus $k_1 = 0$ and $k_2 = \frac{2V_0}{L\sqrt{-\beta}}$. Therefore,

$$\begin{aligned} I_L(t) &= \frac{2V_0}{L\sqrt{-\beta}} e^{\alpha t/2} \sin\left(\frac{\sqrt{-\beta}}{2} t\right) \\ &= \frac{2V_0}{\sqrt{4\frac{L}{C} - R^2}} e^{-\frac{R}{2L}t} \sin\left(t \frac{\sqrt{\frac{4}{LC} - \left(\frac{R}{L}\right)^2}}{2}\right) \\ &= \frac{2V_0}{\sqrt{4\frac{L}{C} - R^2}} e^{-\frac{R}{2L}t} \sin\left(t \frac{\sqrt{4\frac{L}{C} - R^2}}{2L}\right). \end{aligned}$$

(c) Substituting in the appropriate values, we see that

$$\beta = \frac{R^2}{10^{-8}} - \frac{4}{10^{-14}} = 10^8 R^2 - 4 \times 10^{14}.$$

When the system is critically damped, $\beta = 0$, which means that

$$R = \sqrt{4 \times 10^6} = 2000.$$

Therefore, the values of 10, 2000, and 10000 ohms represent underdamped, critically damped, and overdamped systems respectively.

We plot the solutions below using MATLAB.

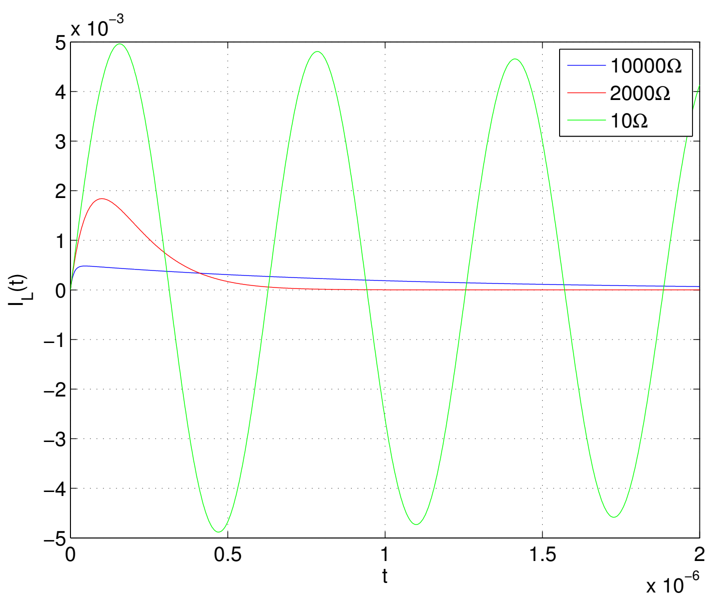


Figure 3: Graph for Problem 5c.

6 Nonhomogeneous RLC Circuit [20 Points]

Solution

(a) We use KVL and KCL to see that $V_S = V_L + V_R + V_C$ and $I_L = I_R = I_C$. Since $I_L = I_C = CV_C'$, we know that $V_C'(0) = 0$. We also know that

$$V_S(0) = V_L(0) + V_R(0) + V_C(0) = LI_L'(0) + RI_L(0) + 0 = LI_L'(0).$$

Thus $I_L'(0) = V_S/L$, since $V_S(t)$ is constant.

We know that $V_L = LI_L' = LI_C' = L(CV_C')' = LCV_C''$, and that $V_R = RI_R = RI_C = RCV_C'$. Therefore, our ODE for V_C is

$$LCV_C'' + RCV_C' + V_C = V_S$$

which we can write as

$$V_C'' + \frac{R}{L}V_C' + \frac{1}{LC}V_C = \frac{V_S}{LC}.$$

The initial conditions are $V_C(0) = V_C'(0) = 0$.

We also know that $V_S' = V_L' + V_R' + V_C'$. Since $V_L' = (LI_L')' = LI_L''$, $V_R' = RI_R' = RI_L'$,, and $V_C' = I_C/C = I_L/C$, our ODE for I_L is

$$LI_L'' + RI_L' + I_L/C = V_S'$$

which we can write as

$$I_L'' + \frac{R}{L}I_L' + \frac{1}{LC}I_L = \frac{V_S'}{L}.$$

The initial conditions are $I_L(0) = 0$ and $I_L'(0) = V_S/L$.

(b) The characteristic equation is

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0.$$

Thus using the quadratic formula,

$$\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}.$$

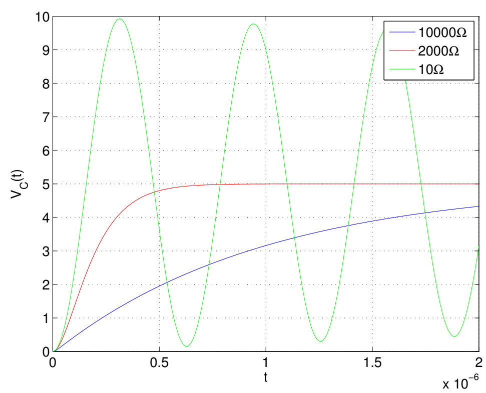
(c) The system will be overdamped if

$$\left(\frac{R}{L}\right)^2 - \frac{4}{LC} > 0.$$

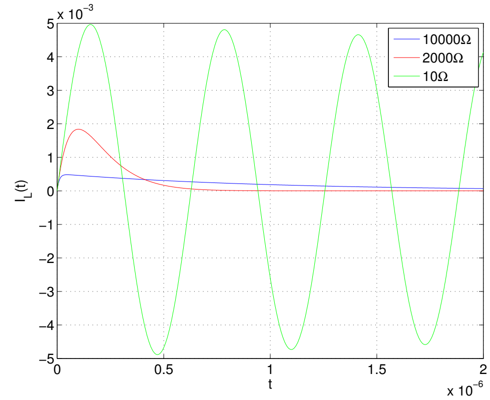
Solving this, we get

$$R > 2\sqrt{\frac{L}{C}} = 2\sqrt{10^6} = 2000.$$

Then, we know that if $R < 2000$ the system will be underdamped, and if $R = 2000$ then the system will be critically damped.



(a) Graph for $V_C(t)$ in Problem 6d.



(b) Graph for $I_L(t)$ in Problem 6d.

(d) In MATLAB we can use the `ode45` function to plot V_C and I_L directly from the ODE. Using this, we can create the plots shown in Figures 4a and 4b.

Notice that since V_S is constant, $V_S' = 0$. Additionally, $I_L'(0) = V_S/L$, where V_S is the same as V_0 from the previous problem. Therefore, our plot for $I_L(t)$ is identical to the one in Problem 5c.

7 Chapter 2 Review Problem 26 [10 Points]

This is a 2nd-order R-L-C circuit with a sinusoidal driving function. Follow a methodology similar to what you did in problem 6, but use the component values for this problem from Kreysig instead of the ones you used in problem 6. Assume that the initial conditions on $V_C(t)$ and $I_L(t)$ are: $V_C(0) = 0\,\text{V}$; $I_L(0) = 0\,\text{A}$ for $t < 0\,\text{s}$ and that $E(t)$ is turned on right at $t = 0\,\text{s}$. Solve for $I_L(t)$ as the problem asks.

Find the current in the RLC-circuit in the figure below when $R = 40\,\Omega$, $L = 0.4\,\text{H}$, $C = 10^{-4}\,\text{F}$, $E = 220 \sin 314t\,\text{V}$ (50 cycles/sec).

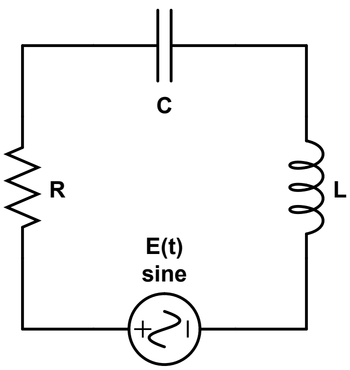


Figure 4: Figure for Problem 7.

Solution

Our circuit laws tell us that $E(t) = V_R + V_C + V_L$ and $I_R = I_C = I_L$. We know that $V_L = LI_L'$, $V_R = RI_R = RI_L$, and $I_C = CV_C'$. If we take the derivative of the equation we get from applying KVL, then we have

$$\begin{aligned} E'(t) &= V_R' + V_C' + V_L' \\ &= (RI_L)' + I_C/C + (LI_L')' \\ &= RI_L' + I_L/C + LI_L''. \end{aligned}$$

Therefore, our ODE is

$$I_L'' + \frac{R}{L}I_L' + \frac{1}{LC}I_L = \frac{1}{L}E'(t).$$

We find the derivative of $E(t)$:

$$E'(t) = 220(314)\cos(314t) = 69080\cos(314t).$$

Substituting the given values into the equation, we have

$$I_L'' + 100I_L' + 25000I_L = 172700\cos(314t).$$

We know that $E(0) = V_R(0) + V_C(0) + V_L(0) = LI_L'(0)$. Since $E(0) = 0$, $I_L'(0) = 0$. Thus our initial conditions are $I_L(0) = I_L'(0) = 0$.

Our characteristic equation is $\lambda^2 + 100\lambda + 25000 = 0$. Then,

$$\lambda = \frac{-100 \pm \sqrt{10000 - 100000}}{2} = -50 \pm 150j.$$

Then, our homogeneous solution is of the form

$$I_{L_h}(t) = e^{-50t}(k_1 \cos(150t) + k_2 \sin(150t)).$$

To find the particular solution, we let $I_{L_p}(t) = K \cos(314t) + M \sin(314t)$ and use the following system of equations:

$$(25000 - 314^2)K + 31400M = 172700$$
$$-31400K + (25000 - 314^2)M = 0$$

The solutions are approximately $K = -1.985$ and $M = 0.847$.

We can now solve for our initial conditions:

$$I_L(0) = k_1 - 1.985 = 0 I_L'(0) = -50k_1 + 150k_2 + 314(0.847) = 0$$

This has the solution $k_1 = 1.985$, $k_2 = -2.435$. Therefore,

$$I_L(t) = e^{-50t}(1.985 \cos(150t) - 2.435 \sin(150t)) - 1.985 \cos(314t) + 0.847 \sin(314t).$$